

8 Feedback Linearization

Key points

- Feedback linearization = ways of transforming original system models into equivalent models of a simpler form.
- Completely different from conventional (Jacobian) linearization, because feedback linearization is achieved by exact state transformation and feedback, rather than by linear approximations of the dynamics.
- Input-Output, Input-State
- Internal dynamics, zero dynamics, linearized zero dynamics
- Jacobi's identity, the theorem of Frobenius
- MIMO feedback linearization is also possible.

Feedback linearization is an approach to nonlinear control design that has attracted lots of research in recent years. The central idea is to algebraically transform nonlinear systems dynamics into (fully or partly) linear ones, so that linear control techniques can be applied.

This differs entirely from conventional (Jacobian) linearization, because feedback linearization is achieved by exact state transformation and feedback, rather than by linear approximations of the dynamics.

The basic idea of simplifying the form of a system by choosing a different state representation is not completely unfamiliar; rather it is similar to the choice of reference frames or coordinate systems in mechanics.

Feedback linearization = ways of transforming original system models into equivalent models of a simpler form.

Applications: helicopters, high-performance aircraft, industrial robots, biomedical devices, vehicle control.
Warning: there are a number of shortcomings and limitations associated with the feedback linearization approach. These problems are very much topics of current research.

References: Sastry, Slotine and Li, Isidori, Nijmeijer and van der Schaft

Terminology

Feedback Linearization

A "catch-all" term which refers to control techniques where the input is used to linearize all or part of the system's differential equations.

Input/Output Linearization

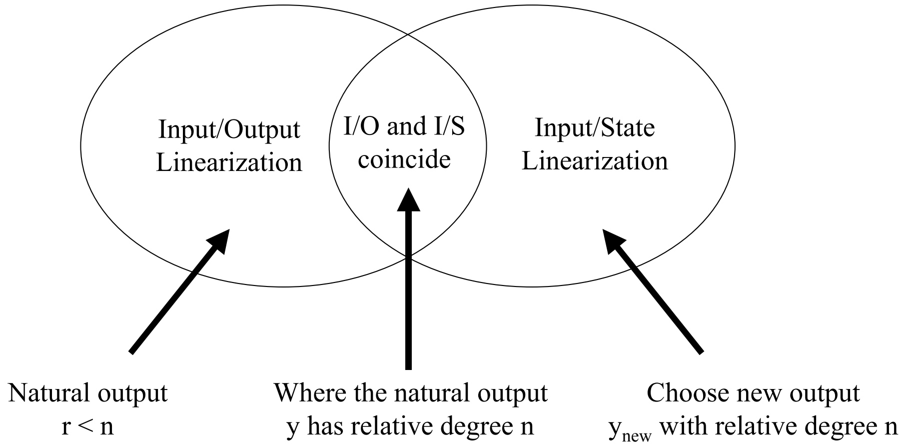
A control technique where the output y of the dynamic system is differentiated until the physical input u appears in the r^{th} derivative of y. Then u is chosen to yield a transfer function from the "synthetic input", v, to the output y which is:

$$\frac{Y(s)}{V(s)} = \frac{1}{s^r}$$

If r, the relative degree, is less than n, the order of the system, then there will be internal dynamics. If r = n, then I/O and I/S linearizations are the same.

Input/State Linearization

A control technique where some new output $y_{new} = h_{new}(x)$ is chosen so that with respect to y_{new}, the relative degree of the system is n. Then the design procedure using this new output y_{new} is the same as for I/O linearization.



SISO Systems

Consider a SISO nonlinear system:

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\end{aligned}$$

Here, u and y are scalars.

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} L^1_f h + L_g(h)u = L^1_f h$$

If $L_g h = 0$, we keep taking derivatives of y until the output u appears. If the output doesn't appear, then u does not affect the output! (Big difficulties ahead).

$$\ddot{y} = L^2_f h + L_g(L^1_f h)u = L^2_f h \quad \text{If } L_g(L^1_f h) = 0\text{, we keep going.}$$

We end up with the following set of equalities:

$$\begin{aligned}
y &= h(x) = L^0_f h \\
\dot{y} &= L^1_f h + L_g(h)u = L^1_f h \quad \text{with } L_g h = 0 \\
\ddot{y} &= L^2_f h + L_g(L^1_f h)u = L^2_f h \quad \text{with } L_g(L^1_f h) = 0 \\
&\dots \\
y^{(r)} &= L^r_f h + L_g(L^{r-1}_f h)u = v \quad \text{with } L_g(L^{r-1}_f h) \neq 0
\end{aligned}$$

The letter r designates the <u>relative degree</u> of y=h(x) iff:

$$L_g\left(L^{r-1}_f(h)\right) \neq 0$$

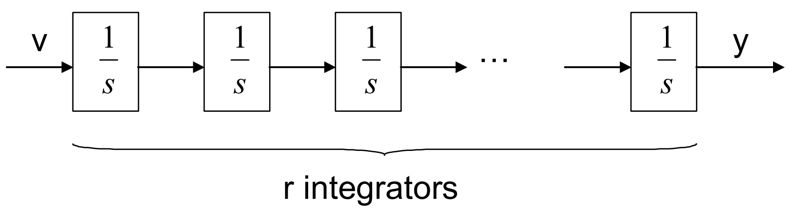
That is, r is the smallest integer for which the coefficient of u is non-zero over the space where we want to control the system.

Let's set:

$$\alpha(x) = L_f^r(h)$$
$$\beta(x) = L_g(L_f^{r-1}(h))$$

Then $y^{(r)} = L_f^r h + L_g(L_f^{r-1}h)u = \alpha(x) + \beta(x)u \equiv v(x)$, where $\beta(x) \neq 0$

v(x) is called the <u>synthetic input</u> or synthetic control. $y^{(r)}=v$



We have an r-integrator linear system, of the form: $\dfrac{Y(s)}{V(s)} = \dfrac{1}{s^r}$.

We can now design a controller for this system, using any linear controller design method. We have $v = \alpha + \beta u$. The controller that is implemented is obtained through:

$$u = \frac{1}{\beta(x)}\left[-\alpha(x) + v\right]$$

Any linear method can be used to design v. For example,

$$v = -\sum_{k=0}^{r-1} c_k L_f^k(h) = -c_0 y - c_1 \dot{y} - c_2 \ddot{y} ...$$

$$\Rightarrow y^{(r)} + c_{r-1} y^{(r-1)} + ... + c_0 y = 0$$

<u>Problems with this approach:</u>

1. Requires a perfect model, with perfect derivatives (one can anticipate robustness problems).
2. If the goal is $y \to y_d(t)$, $v = -c_0(y - yd) - ... - c_{r-1}(y^{(r-1)} - y_d^{(r-1)})$.

   If $x \in \Re^{20}$, and r = 2, there are 18 states for which we don't know what is happening. That is, if $r < n$, we have <u>internal dynamics</u>.

Note: There is an <u>ad-hoc approach to the robustness problem</u>, by setting:

$$v = -\sum_{k=0}^{r-1} c_k L_f^k(h) + K_c\left[(y_d - y) + \frac{1}{\tau}\int_0^{\tau}(y_d - y)d\tau\right]$$

Here the first term in the expression is the standard feedback linearization term, and the second term is tuned online for robustness.

Internal Dynamics

Assume r<n $\Rightarrow$ there are some internal dynamics

$$z \equiv \begin{bmatrix} z_1 \\ z_2 \\ \ldots \\ z_r \end{bmatrix} \text{ where } \begin{array}{l} z_1 \equiv y = L_f^0 h \\ z_2 = \dot{y} = L_f^1 h \\ \ldots \\ z_r = y^{(r-1)} = L_f^{r-1} h \end{array}$$

So we can write:

$$\boxed{\dot{z} = Az + Bv}$$

where A and B are in controllable canonical form, that is:

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ 1 \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \end{bmatrix} z$$

$$\text{where } A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ 1 \end{bmatrix}$$

We define:

$$x = \begin{bmatrix} z \\ \xi \end{bmatrix}$$ where z is rx1 and ξ is (n-r)x1. ($z \in \Re^r, \xi \in \Re^{n-r}$).

The <u>normal forms theorem</u> tells us that there exists an ξ such that:

$$\boxed{\dot{\xi} = \psi(z,\xi)}$$

Note that the internal dynamics are not a function of u.

So we have:

$$\boxed{\begin{cases} \dot{z} = Az + Bv \\ \dot{\xi} = \psi(z,\xi) \end{cases}}$$

The ξ equation represents "<u>internal dynamics</u>"; these are not observable because z does not depend on ξ at all ⇒ "internal", and hard to analyze!

We want to analyze the zero dynamics. The system is difficult to analyze. Oftentimes, to make our lives easier, we analyze the so-called "<u>zero dynamics</u>":

$$\boxed{\dot{\xi} = \psi(0,\xi)}$$

and in most cases we even look at the "<u>linearized zero dynamics</u>".

$$J = \left.\frac{\partial \psi}{\partial \xi}\right|_0$$ and we look at the eigenvalues of J.

If these are well behaved, perhaps the nonlinear dynamics might be well-behaved. If these are not well behaved, the control may not be acceptable!

<u>For linear systems:</u>

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

We have: $$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B$$

The eigenvalues of the zero dynamics are the zeroes of H(s). Therefore if the zeroes of H(s) are non-minimum phase (in the right-half plane) then the zero dynamics are unstable.

♦

By analogy, for nonlinear systems: if $\dot{\xi} = \psi(0,\xi)$ is unstable, then the system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

is called a non-minimum phase nonlinear system.

Input/Output Linearization

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

- **Procedure**

a) Differentiate y until u appears in one of the equations for the derivatives of y

$$\begin{gathered} \dot{y} \\ \ddot{y} \\ \ldots \\ y^{(r)} = \alpha(x) + \beta(x)u \end{gathered}$$

after r steps, u appears

b) Choose u to give $y^{(r)}=v$, where v is the synthetic input

$$u = \frac{1}{\beta(x)}\left[-\alpha(x) + v\right]$$

c) Then the system has the form: $\dfrac{Y(s)}{V(s)} = \dfrac{1}{s^r}$

Design a linear control law for this r-integrator liner system.

d) Check internal dynamics.

- **Example**

Oral exam question

Design an I/O linearizing controller so that y → 0 for the plant:

$$\begin{cases} \dot{x}_1 = x_2 + x_1^3 + u \\ \dot{x}_2 = -u \end{cases}$$
$$y = h(x) = x_1$$

Follow steps:

a) $\dot{y} = \dot{x}_1 = x_2 + x_1^3 + u$ u appears ⇒ r = 1

b) Choose u so that $\dot{y} = v = x_2 + x_1^3 + u$

$$\Rightarrow \; u = -x_2 - x_1^3 + v$$

In our case, $\alpha(x) = x_1^3 + x_2$ and $\beta(x) = 1$.

c) Choose a control law for the r-integrator system, for example proportional control

Goal: to send y to zero exponentially

$$\Rightarrow \; v = -K_p(y - y_{des}) = -K_p y$$ since $y_{des} = 0$

d) Check internal dynamics:

Closed loop system:

$$\dot{x}_1 = v = -K_p x_1$$
$$\dot{x}_2 = -u = -(-x_1^3 - x_2 + v) = -(-x_1^3 - x_2 - K_p x_1) = x_1^3 + K_p x_1 + x_2$$

If $x_1 \to 0$ as desired, x_2 is governed by $\dot{x}_2 = x_2$
⇒ ***Unstable internal dynamics!***

There are two possible approaches when faced with this problem:

- Try and redefine the output: $y=h(x_1,x_2)$
- Try to linearize the entire system/space ⇒ Input/State Linearization

♦

Input/State Linearization (SISO Systems)

$$\dot{x} = f(x) + g(x)u$$

Question: does there exist a transformation $\phi(x)$ such that the transformed system is linear?

Define the transformed states:

$$z \equiv \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_n \end{bmatrix}$$

I want to find $\phi(x)$ such that $\dot{z} = Az + Bv$ where $v \in \Re$, with:

- v=v(x,u) is the synthetic control
- the system is in Brunowski (controllable) form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}$$

A is nxn and B is nx1.

We want a 1 to 1 correspondence between z and x such that:

$$z \equiv \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_n \end{bmatrix} \Leftrightarrow x \equiv \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}$$

Question: does there exist an output $y=z_1(x)$ such that y has relative degree n?

$\dot{z}_1 = L_f^1 h + L_g(h)u = L_f^1 h$ with $L_g h = 0$

Let $z_2 \equiv L_f^1(z_1)$

Then: $L_g(L_f^1(z_1)) = 0$

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\dots \\
\dot{z}_n &= L_f^n(z_1) + L_g(L_f^{n-1}(z_1))u \equiv v
\end{aligned}$$

⇒ does there exist a scalar $z_1(x)$ such that:

$$L_g\left(L_f^k(h)\right) = 0 \text{ for k = 1,…,n-2}$$
$$\text{And } L_g\left(L_f^{n-1}(h)\right) \neq 0 \text{ ?}$$

$$z \equiv \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_r \end{bmatrix} = \begin{bmatrix} L_f^0(z_1) \\ L_f^1(z_1) \\ \dots \\ L_f^{n-1}(z_1) \end{bmatrix}$$

⇒ is there a test?

$$\dot{x} = f(x) + g(x)u$$

so the test should depend on f and g.

Jacobi's identity

Carl Gustav Jacob Jacobi

Born: 10 Dec 1804 in Potsdam, Prussia (now Germany)
Died: 18 Feb 1851 in Berlin, Germany

Famous for his work on:

- Orbits and gravitation
- General relativity
- Matrices and determinants

Jacobi's Identity

A convenient relationship (S+L) is called "Jacobi's identity".

$$\boxed{L_{ad_f}g(h) = L_f(L_g(h)) - L_g(L_f(h))}$$

Remember:

$$ad_f^i g \equiv \left[f, ad_f^{i-1} g\right] \text{ and } ad_f g = [f, g]$$

This identity allows us to keep the conditions in first order in z_1

⇒ Trod through messy algebra

- *For k = 0*

$$L_g(L_f^0(z_1)) = 0 \Rightarrow L_g(z_1) = 0$$

$$\frac{\partial z_1}{\partial x_1}.g_1 + \frac{\partial z_2}{\partial x_2}.g_2 + ... + \frac{\partial z_n}{\partial x_n}.g_n = 0 \quad \text{(first order)}$$

- *For k = 1*

$$L_g(L_f^1(z_1)) = 0$$

$$L_g\left(\frac{\partial z_1}{\partial x_1}.f_1 + \frac{\partial z_2}{\partial x_2}.f_2 + ... + \frac{\partial z_n}{\partial x_n}.f_n\right) = 0$$

$$\nabla\left(\frac{\partial z_1}{\partial x_1}.f_1 + \frac{\partial z_2}{\partial x_2}.f_2 + ... + \frac{\partial z_n}{\partial x_n}.f_n\right).g = 0$$

$$\Rightarrow \frac{\partial^2 z_1}{\partial x_1^2} + \ldots \quad \Rightarrow 2^{\text{nd}} \text{ order (gradient)}$$

Things get messy, but by repeated use of Jacobi's identity (see Slotine and Li), we have:

$$\boxed{L_g\left(L_f^k(z_1)\right) = 0 \quad \text{for } k \in [0, n-2] \quad \Leftrightarrow L_{ad_f^k} g(z_1) = 0 \quad \text{for } k \in [0, n-2] \qquad (*)}$$

The two conditions above are equivalent. Evaluating the second half:

$$L_{ad_f^k} g(z_1) = 0 \Leftrightarrow \nabla z_1 . \left[g, ad_f g, \ldots, ad_f^{n-2} g\right] = 0$$

This leads to conditions of the type:

$$\nabla z_1 . g = 0 \Rightarrow \frac{\partial z_1}{\partial x_1} . g_1 + \frac{\partial z_2}{\partial x_2} . g_2 + \ldots + \frac{\partial z_n}{\partial x_n} . g_n = 0$$

$$\nabla z_1 . ad_f g = 0 \Rightarrow \frac{\partial z_1}{\partial x_1} . (\ldots) + \frac{\partial z_2}{\partial x_2} . (\ldots) + \ldots + \frac{\partial z_n}{\partial x_n} . (\ldots) = 0$$

The Theorem of Frobenius

<u>Ferdinand Georg Frobenius:</u>

Born: 26 Oct 1849 in Berlin-Charlottenburg, Prussia (now Germany)

Died: 3 Aug 1917 in Berlin, Germany

Famous for his work on:
- Group theory
- Fundamental theorem of algebra
- Matrices and determinants

Theorem of Frobenius:

A solution to the set of partial differential equations $L_{ad_f^k} g(z_1) = 0$ for $k \in [0, n-2]$ exists if and only if:

a) $[g, ad_f g, ..., ad_f^{n-1} g]$ has rank n

b) $[g, ad_f g, ..., ad_f^{n-2} g]$ is involutive

♦

Definition of "involutive":

A linear independent set of vectors (f_1, …, f_m) is involutive if:

$$[f_i, f_j] = \sum_{k=1}^{m} \alpha_{ijk}(x) f_k(x) \quad \forall (i,j) \in N^2$$

♦

i.e. when you take Lie brackets you don't generate new vectors.

Note: this is VERY hard to do.
Reference: George Myers at NASA Ames, in the context of helicopter control.

Example: (same as above)

$$\begin{cases} \dot{x}_1 = x_2 + x_1^3 + u \\ \dot{x}_2 = -u \end{cases}$$

Question: does there exist a scalar $z_1(x_1,x_2)$ such that the relative degree be 2?

$$f = \begin{bmatrix} x_2 + x_1^3 \\ 0 \end{bmatrix} \qquad g = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

This will be true if:

a) (g, [f, g]) has rank 2

b) g is involutive (any Lie bracket on g is zero → OK)

Setting stuff up to look at (a):

$$(g,[f,g]) = \begin{bmatrix} -1 & -3x_1^2+1 \\ 1 & 0 \end{bmatrix}$$

Note: $[f,g] = \dfrac{\partial g}{\partial x}.f - \dfrac{\partial f}{\partial x}.g = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 3x_1^2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3x_1^2+1 \\ 0 \end{bmatrix}$

$x_1 = \pm\sqrt{3}/3$ looks dangerous

Question: how do we find z1?

We get a list of conditions:

- $\nabla z_1.g = 0 \Rightarrow \begin{bmatrix} \dfrac{\partial z_1}{\partial x_1} & \dfrac{\partial z_2}{\partial x_2} \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$

  $$\Rightarrow \frac{\partial z_1}{\partial x_1} = \frac{\partial z_2}{\partial x_2} \qquad \Rightarrow z_1 = x_1 + x_2$$

- $\nabla z_1.ad_f g = 0$ (automatically)

So let's trod through and check:

$z_1 = x_1 + x_2$

$\dot{z}_1 = \dot{x}_1 + \dot{x}_2 = x_2 + x_1^3 = z_2$ (good that u doesn't appear, or r=1!)

$\ddot{z}_1 = \dot{z}_2 = \dot{x}_2 + 3x_1^2\dot{x}_1 = 3x_1^2(x_2 + x_1^3) + (3x_1^2 - 1).u$ (u appears! (good))

Question: if you want y=x_1 like in the original problem:

Define $\ddot{z}_1 = v$, $\quad z_1 = x_1 + x_2$

Hope the problem is far away from $x_1 = \pm\sqrt{3}/3$

Let $v = -c_1\dot{z}_1 - c_2(z_1 - z_{1d})$

$$\Rightarrow \ddot{z}_1 + c_1\dot{z}_1 + c_2 z_1 = c_2 z_{1d}$$

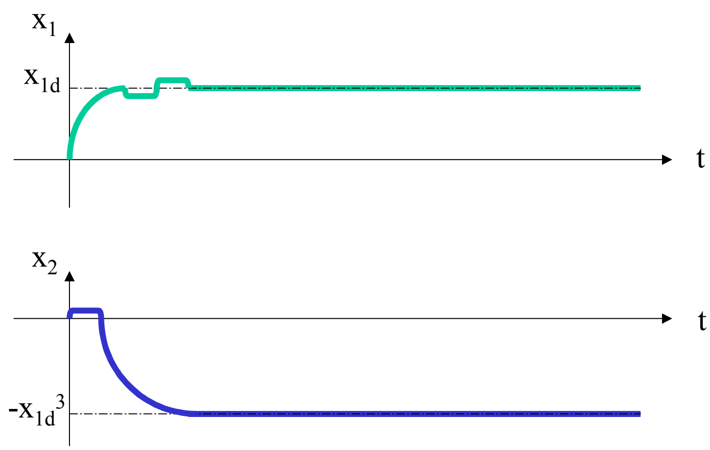
$$\Rightarrow z_1 \rightarrow z_{1d}$$

Question: How to pick z_{1d}?

$$z_1 = x_1 + x_2$$

$$z_{1d} = x_{1d} + x_{2d}$$

We want: $x_{2d} \equiv -x_{1d}^3$ for $\dot{z}_1 = 0 = x_2 + x_1^3$



Feedback Linearization for MIMO Nonlinear Systems

Consider a "square" system (where the number of inputs is equal to the number of outputs = n)

$$\begin{cases} \dot{x} = f(x) + \sum_{i=1}^{m} g_i . u_i \\ y = [h_1, \ldots h_m]^T \end{cases}$$

$$\dot{y}_k = L_f(h_k) + \sum_{i=1}^{m} L_{g_i}(h_k) u_i$$

Let $\boldsymbol{r_k}$, the <u>relative degree</u>, be defined as the relative degree of each output, i.e.

For some i, $L_{g_i}\left(L_f^{r_k-1}(h_k)\right) \neq 0$

Let J(x) be an mxm matrix such that:

$$J(x) \equiv \begin{bmatrix} L_{g_1}(L_f^{r_1-1}(h_1)) & \dots & L_{g_m}(L_f^{r_1-1}(h_1)) \\ \dots & \dots & \dots \\ L_{g_1}(L_f^{r_m-1}(h_m)) & \dots & L_{g_m}(L_f^{r_m-1}(h_m)) \end{bmatrix}$$

<u>J(x) is called the invertibility or decoupling matrix.</u>

We will assume that J(x) is non-singular.

Let:

$$y^r \equiv \begin{bmatrix} \dfrac{d^{r_1} y_1}{dt^{r_1}} \\ \dots \\ \dfrac{d^{r_m} y_m}{dt^{r_m}} \end{bmatrix}$$ where y^r is an mx1 vector

$$l(x) = \begin{bmatrix} L_f^{r_1}(h_1) \\ \dots \\ L_f^{r_m}(h_m) \end{bmatrix}$$

Then we have:

$$\boxed{y^r \equiv l(x) + J(x).u \equiv v}$$ where v is the synthetic input (v is mx1).

We obtain a decoupled set of equations:

$$\begin{cases} \dfrac{d^{r_1} y_1}{dt^{r_1}} = v_1 \\ \dots \\ \dots \\ \dfrac{d^{r_m} y_m}{dt^{r_m}} = v_m \end{cases} \quad \text{so} \quad y \Leftrightarrow v$$

Design v any way you want to using linear techniques…

$$\boxed{u = J^{-1}(v - l)}$$

Problems:

- Need confidence in the model
- Internal dynamics

Internal Dynamics

The linear subspace has dimension (or relative degree) for the whole system:

$$r_T = \sum_{k=1}^{m} r_k$$

$\Rightarrow$ we have internal dynamics of order n-r_T.

$$\begin{cases} z_1^i = h_i \\ \dot{z}_1^i = z_2^i \\ \dots \\ \dot{z}_{r_i}^i = L_f^{r_i}(h_i) + \sum_1^m L_{g_k}\left(L_f^{r_i - 1}(h_i)\right)u_k \equiv v_i \end{cases}$$

The superscript notation denotes which output we are considering. We have:

$$x = \begin{bmatrix} z_1^1 \\ z_2^1 \\ \dots \\ z_{r_1}^1 \\ z_1^2 \\ \dots \\ \dots \end{bmatrix} \qquad \Rightarrow x = \begin{bmatrix} z^T \\ \xi^T \end{bmatrix}$$ where z^T is rx1, ξ^T is (n-r_T)x1

The representation for x may not be unique!

Can we get a ξ who isn't directly a function of the controls (like for the SISO case)? NO!

$$\boxed{\begin{aligned}\dot{\xi} &= \psi(\xi, z) + P(\xi, z).u \\ \dot{z} &= Az + Bv\end{aligned}}$$

and $\quad y^r \equiv l(x) + J(x).u \equiv v$

Internal dynamics $\quad\Rightarrow$ what is u?
$\Rightarrow$ design v, then solve for u using $u = J^{-1}(v - l)$

The zero dynamics are defined by z = 0.

$$y^r \equiv 0 \;\Rightarrow\; u^* = -J^{-1} l(x)$$

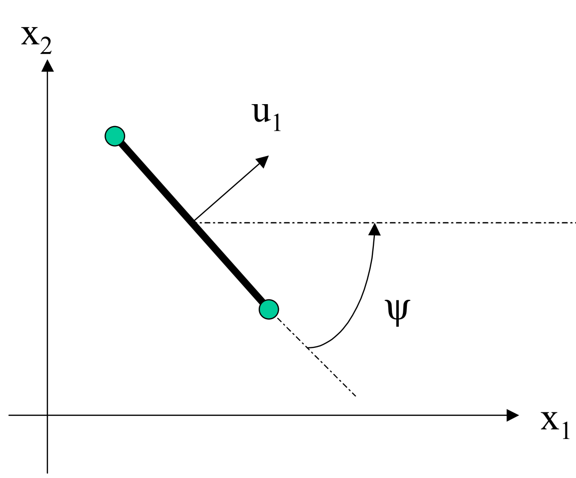
The output is identically equal to zero if we set the control equal to zero (at all times).

Thus the zero dynamics are given by:

$$\dot{\xi} = \psi(\xi,0) - P(\xi,0).J^{-1}(\xi,0) l(\xi,0)$$

Dynamic Extension - Example

References: Slotine and Li
Hauser, PhD Dissertation, UCB, 1989 from which this example is taken



Basically, ψ is the yaw angle of the vehicle, and x_1 and x_2 are the Cartesian locations of the wheels. u_1 is the velocity of the front wheels, in the direction that they are pointing, and u_2 is the steering velocity.

We define our state vector to be:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \psi \end{bmatrix}$$

Our dynamics are:

$$\begin{cases} \dot{x}_1 = \sin\psi u_1 \\ \dot{x}_2 = \cos\psi u_1 \\ \dot{\psi} = u_2 \end{cases}$$

We determined in a previous lecture that the system is controllable (f = 0).

$y_1 \equiv x_1$ and $y_2 \equiv x_2$ are defined as outputs.

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \cos\psi & 0 \\ \sin\psi & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$J(x) = \begin{bmatrix} \cos\psi & 0 \\ \sin\psi & 0 \end{bmatrix}$$ is clearly singular (has rank 1).

Let $u_1 \equiv x_3$, $\dot{u}_1 \equiv \dot{x}_3 = u_3$ where u_3 is the acceleration of the axle

$\Rightarrow$ the state has been extended.

$$\begin{cases} \dot{x}_1 = \cos\psi x_3 \\ \dot{x}_2 = \sin\psi x_3 \\ \dot{x}_3 = \dot{u}_1 = u_3 \\ \dot{\psi} = u_2 \end{cases}$$

$$f = \begin{bmatrix} x_3 \cos\psi \\ x_3 \sin\psi \\ 0 \\ 0 \end{bmatrix} \qquad g_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $\dot{x} = f + g_1 u_3 + g_2 u_2$ in the <u>extended state space</u>

Take $y_1 \equiv x_1$ and $y_2 \equiv x_2$.

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} \cos\psi & -u_1 \sin\psi \\ \sin\psi & u_1 \cos\psi \end{bmatrix} \begin{bmatrix} u_3 \\ u_2 \end{bmatrix}$$

and the new J(x) matrix: $J(x) = \begin{bmatrix} \cos\psi & -u_1 \sin\psi \\ \sin\psi & u_1 \cos\psi \end{bmatrix}$ is non-singular for $u_1 \neq 0$ (as long as the axle is moving).

How does one go about designing a controller for this example?

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \cos\psi & -u_1 \sin\psi \\ \sin\psi & u_1 \cos\psi \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = J(x) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{cases} \ddot{y}_1 = v_1 \\ \ddot{y}_2 = v_2 \end{cases}$$

Given $y_{1d}(t)$, $y_{2d}(t)$:

Let:

$$\begin{cases} v_1 = -c_1 \dot{y}_1 - c_2 (y_1 - y_{1d}) \\ v_2 = -c_3 \dot{y}_2 - c_4 (y_2 - y_{2d}) \end{cases} \Rightarrow \begin{cases} \ddot{y}_1 + c_1 \dot{y}_1 + c_2 y_1 = c_2 y_{1d} \\ \ddot{y}_2 + c_3 \dot{y}_2 + c_4 y_2 = c_4 y_{2d} \end{cases}$$

To obtain the control, u:

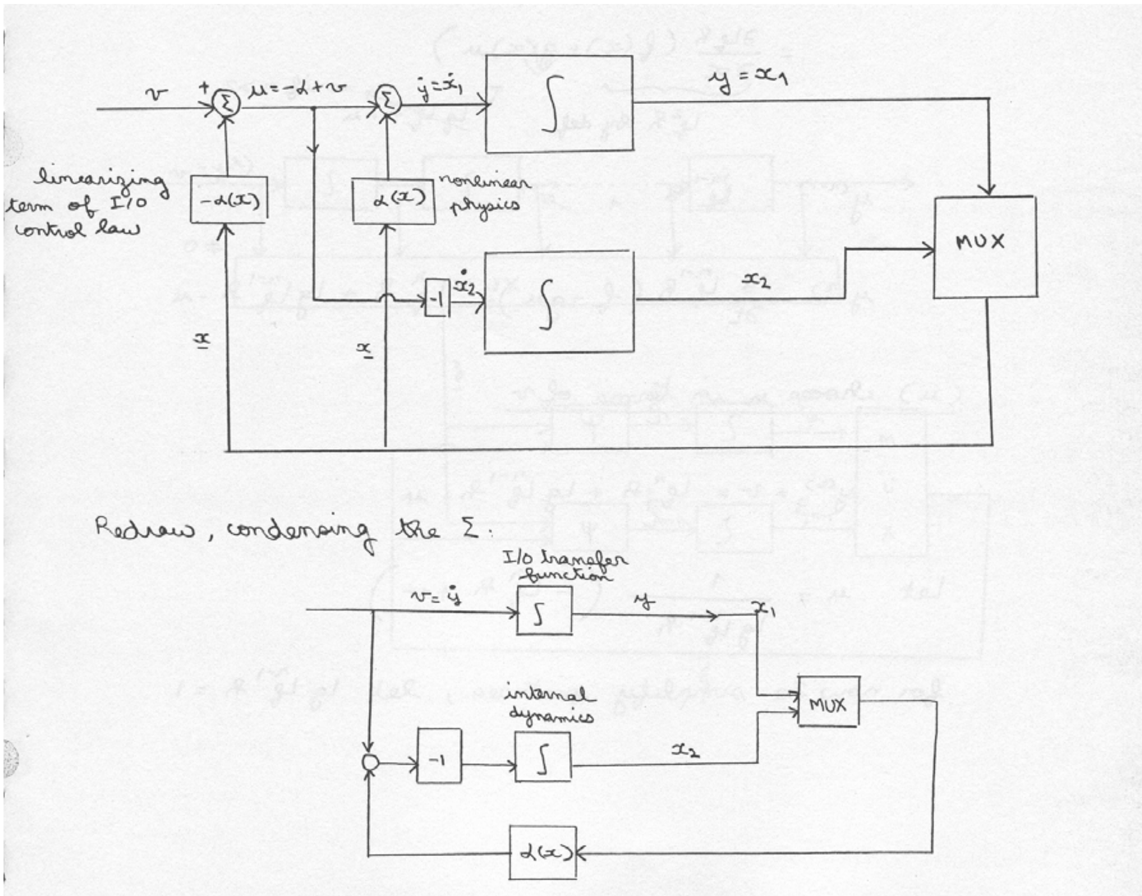
$$\begin{bmatrix} u_3 \\ u_2 \end{bmatrix} = J^{-1}(x) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

and $\dot{u}_1 = u_3 \Rightarrow$ we have a ***dynamic feedback controller*** (the controller has dynamics, not just gains, in it).

♦

Pictures for SISO cases:

- **Picture of I/O system (r = 1)**



- **In general terms**

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

n^{th} order
r = relative degree < n

a) Differentiate:

$$\dot{y} = \dot{h}(x) = \frac{\partial h}{\partial x}.\frac{\partial x}{\partial t} = \frac{\partial h}{\partial x}.(f(x) + g(x)u) = L_f h + L_g h.u \qquad \text{and } L_g h = 0 \text{ if r>1}$$

$\Rightarrow \dot{y} = L_f h$

$$\ddot{y} = \frac{\partial \dot{y}}{\partial t} = \frac{\partial}{\partial t}\left[L_f h\right] = \frac{\partial L_f h}{\partial x}\dot{x} = \frac{\partial L_f h}{\partial x}.\left[f(x) + g(x).u\right] = L_f^2 h + L_g L_f h.u$$

and $L_g L_f h.u = 0$ if r<2

$\Rightarrow \ddot{y} = L_f^2 h$

…

$y^{(r-1)} = L_f^{r-1} h$

$$y^{(r)} = \frac{\partial}{\partial t} L_f^{r-1} h\left[f(x) + g(x)u\right] = L_f^r h + L_g L_f^{r-1} h.u \text{ where } L_g L_f^{r-1} h \neq 0$$

b) Choose u in terms of v

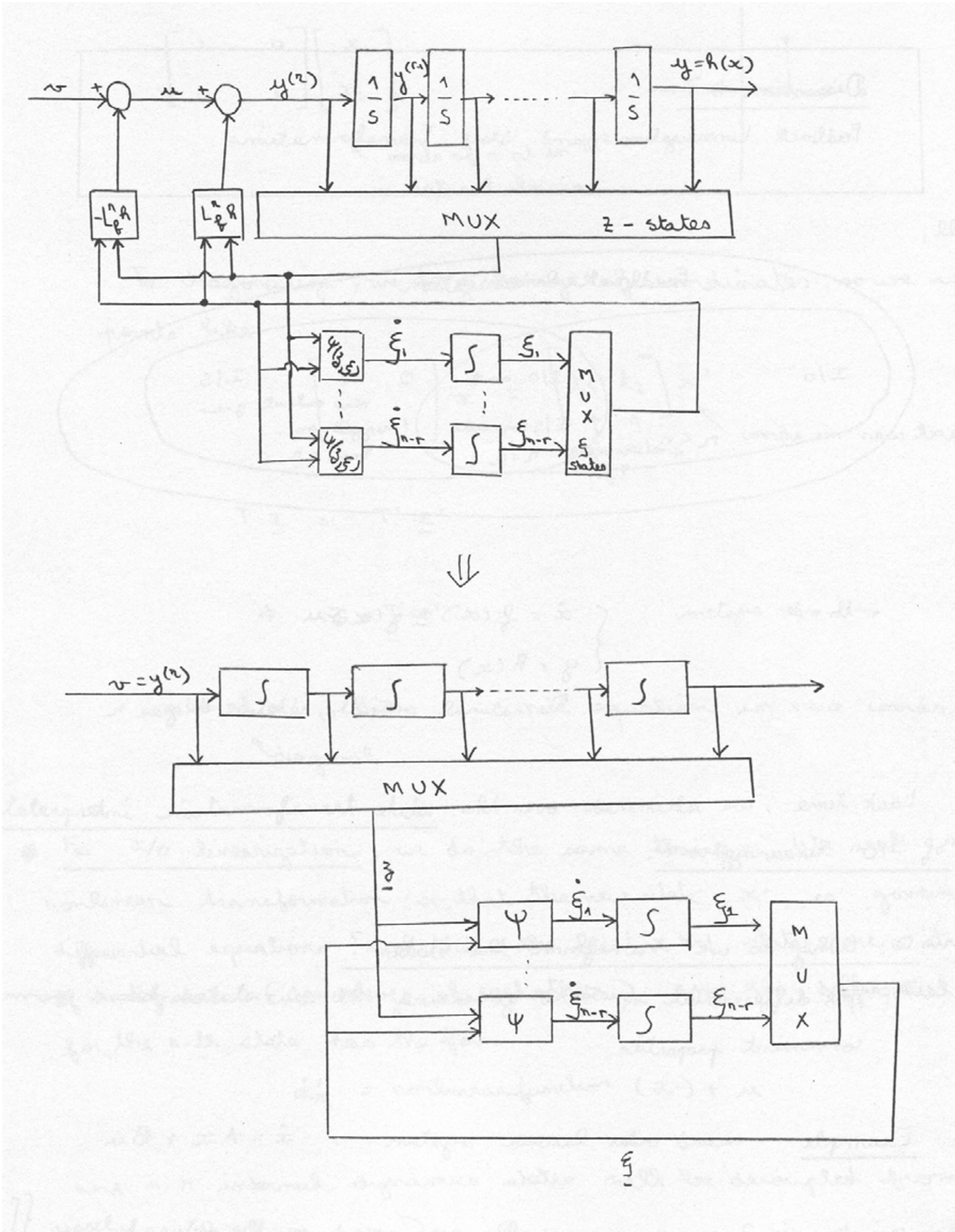
$y^{(r)} = L_f^r h + L_g L_f^{r-1} h.u = v$

Let $u = \dfrac{1}{L_g L_f^{r-1} h}(-L_f^r h + v)$

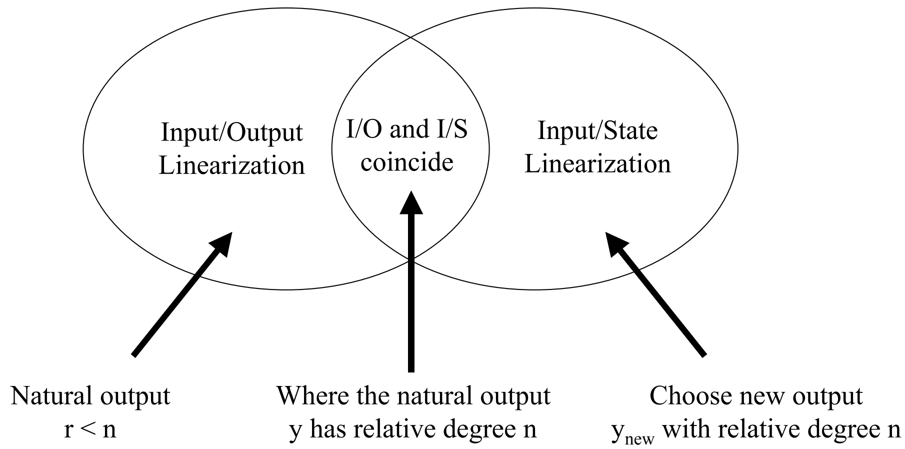
For now, to simplify the pictures, let $L_g L_f^{r-1} h = 1$

c) Choose control law

d) Check internal dynamics


v
u
y^(r)
1/s
y^(r-1)
y = h(x)
MUX
z - states
-L_f^n h
L_f^n h
ψ(z, ξ)
ξ̇_1
ξ_1
ξ̇_{n-r}
ξ_{n-r}
M U X
ξ states
⇓
v = y^(r)
MUX
z
ψ
ξ

Feedback Linearization and State Transformation



$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

We have an n^{th} order system where y is the natural output, with relative degree r.

Previously, we skimmed over the <u>state transformation interpretation of feedback linearization</u>.

<u>Why do we transform the states?</u>

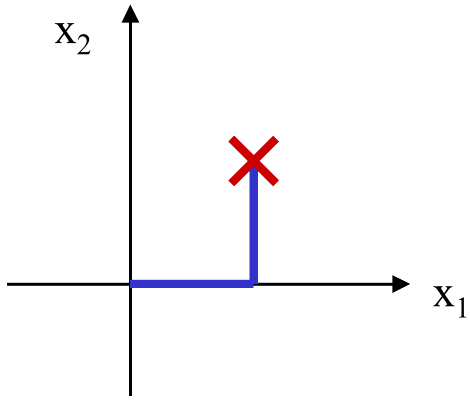
The differential equations governing the new states have some convenient properties.

<u>Example:</u>

Consider a linear system $\dot{x} = Ax + Bu$

The points in 2-space are usually expressed in the natural basis: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

So when we write "x", we mean a point in 2-space that is gotten to from the origin by doing:



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where (x_1, x_2) are the coordinates of a point in $\Re^2$ in the natural basis.

To diagonalize the system, we do a change of coordinates, so we express points like:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} t_1 & t_2 \end{bmatrix} x'$$

where t_1 and t_2 are the eigenvectors of A and x' represents the coordinates in the new basis.

$$\Rightarrow Tx = T'x' \Rightarrow x = T'x'$$

So we get a nice equation in the new coordinates:

$$\dot{x}' = \Lambda x' + B'u$$

where Λ is diagonal.

<u>For I/O linearization, we do the same kind of thing:</u>

We seek some nonlinear transformation so that the new state, x', is governed by differential equations such that the first r-1 states are a string of integrators (derivatives of each other), and the differential equation for the r^{th} state has the form:

$$\dot{x}'_r = \text{nonlinear function(x)} + u$$

and n-r internal dynamics states will be decoupled from u (this is a matter of convenience).

So we have: x'=T(x) where T is nonlinear.

$$x' = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \\ \ldots \\ T_n(x) \end{bmatrix}$$

Let's enforce the above properties:

$$\dot{x}' = \begin{bmatrix} \dot{z}_1 \\ \ldots \\ \ldots \\ \dot{z}_r \\ \dot{\xi}_1 \\ \ldots \\ \dot{\xi}_{n-r} \end{bmatrix} = \begin{bmatrix} z_2 \\ \ldots \\ z_r \\ nonlin.fn(x) + u \\ \Phi_1(z,\xi) \\ \ldots \\ \Phi_{n-r}(z,\xi) \end{bmatrix}$$

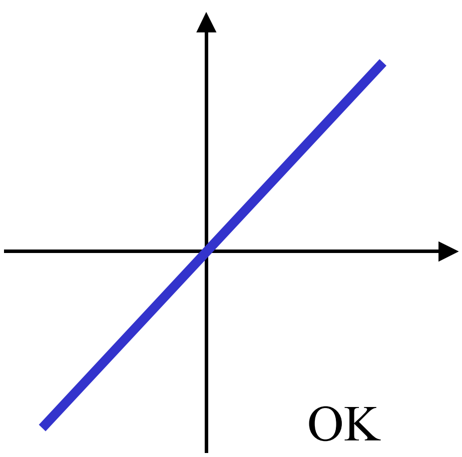
We know how to choose $T_1(x)$ through $T_r(x)$. They are just $y, \dot{y}, \ddot{y}, \ldots$ etc…

<u>How do we choose the $T_{r+1}(x)$ through $T_n(x)$?</u>

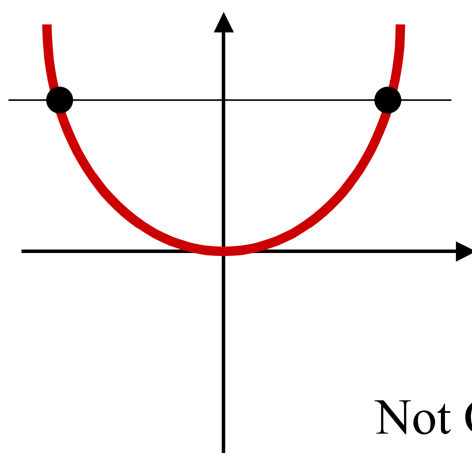
These transformations need to be chosen so that:

1. The transformation T(x) is a diffeomorphism:

    - One to one transformation



OK



Not OK

    - T(x) is continuous
    - $T^{-1}(x')$ is continuous

- o Also $\frac{\partial T}{\partial x}$ and $\frac{\partial T^{-1}}{\partial x'}$ must exist and be continuous

2. The ξ states should have no direct dependence on the input u.

<u>Example (from HW)</u>

$$x' = \begin{bmatrix} z \\ \xi \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$$

We know that $T_1(x_1,x_2) = y = x_2$.

<u>What about $T_2(x_1,x_2)$?</u>

Choose $T_2(x_1,x_2)$ to satisfy the above conditions. Let's start with condition 2, u does not appear in the equation for $\dot{\xi}$.

$$\dot{\xi} = fn(z,\xi)$$

$$\begin{aligned} \dot{\xi} &= \begin{bmatrix} \frac{\partial T_2(x_1,x_2)}{\partial x_1} & \frac{\partial T_2(x_1,x_2)}{\partial x_2} \end{bmatrix} \dot{x} \\ &= \begin{bmatrix} \frac{\partial T_2(x_1,x_2)}{\partial x_1} & \frac{\partial T_2(x_1,x_2)}{\partial x_2} \end{bmatrix} \left(\begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \right) \end{aligned}$$

We are only concerned about the second term. To eliminate the dependence in u, we must have:

$$\begin{bmatrix} \frac{\partial T_2(x_1,x_2)}{\partial x_1} & \frac{\partial T_2(x_1,x_2)}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \frac{\partial T_2(x_1,x_2)}{\partial x_2} = 0$$

$$\Rightarrow T_2(x_1,x_2) = T_2(x_1)$$

(T_2 should not depend on x_2).

An obvious answer is : $T_2(x_1,x_2) = x_1$. Then, we would have:

$$x' = \begin{bmatrix} z \\ \xi \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

Is this a diffeomorphism? Obviously yes.

Note that $T_2(x_1) = x_1^3$ works also.

What about $T_2(x_1) = x_1^2$? (NO – violates one-to-one transformation part of the conditions for a proper diffeomorphism).